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On Spătaru's Extension of the Hsu-Robbins-Erdős Law of Large Numbers*

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The Hsu-Robbins-Erdős law of large numbers (1947, 1949) states that if $X_1, X_2, \ldots$ are independent identically distributed random variables and $S_n = X_1 + \cdots + X_n$, then

$$\sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) < \infty$$

for every $\varepsilon > 0$ if and only if $E[X_1^2] < \infty$ and $E[X_1] = 0$. Under some auxiliary conditions, Spătaru (1994) extended this to the case where the X_n are no longer identically distributed, but rather their distributions come from a finite set of distributions. We improve Spătaru's conditions, and present a counterexample to a conjecture of his.

1. INTRODUCTION AND MAIN RESULTS

In the case that we will consider in the present paper, $X_1, X_2, \ldots$ will denote independent random variables such that the set of the distributions of $X_1, X_2, \ldots$ is finite. Hence, there will exist a finite collection of random variables $Y_1, \ldots, Y_p$ with the property that for each $n \in \mathbb{Z}^+$ there is an $i = i(n)$ such that X_n and Y_i have the same distribution. Let $S_n = X_1 + \cdots + X_n$. In the terminology of Durrett, Kesten, and Lawler [14], the S_n will be called a *finitely inhomogeneous random walk*. See Spătaru [42] for further background on the S_n; other results on the S_n can be found in [14, 26].

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We are interested in the connection between the assertion that

$$\sum_{n=1}^{\infty} P\big(|S_n| \geq \varepsilon n\big) < \infty, \qquad \forall \varepsilon > 0, \tag{1}$$

and the conditions

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} P\big(|X_k| \geq n\big) < \infty \tag{2}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E\big[X_k 1_{\{|X_k| < n\}}\big] = 0. \tag{3}$$

In the terminology of Hsu and Robbins [22], expression (1) asserts that S_n/n *converges completely* to 0.

The Hsu-Robbins-Erdős law of large numbers [22, 15, 16] then says that if the X_n are identically distributed (i.e., if $p = 1$ in the finitely inhomogeneous random walk setting), then (1) holds if and only if both (2) and (3) hold.

Spătaru [42] showed that, in the finitely inhomogeneous random walk case, (1) always implies both (2) and (3), while, conversely, if both (2) and (3) hold, and both of the auxiliary Conditions **A** and **B**, below, also hold, then (1) holds. He also conjectured that even without the auxiliary conditions, (2) and (3) still imply (1). As Theorem 2, below, will show, this conjecture is false. We shall also prove that Spătaru's result holds under some other weaker auxiliary conditions. But before we proceed with this, let us first review some of the literature which grew out of the Hsu-Robbins-Erdős result.

There are indeed many generalizations of the Hsu-Robbins-Erdős law of large numbers in the literature; for an extensive bibliography and brief discussion, see [35]. We mention below only some of the many works on the subject.

To generalize the Hsu-Robbins-Erdős law of large numbers, one could, for instance, try to replace the sum in (1) by a weighted sum, and also one could try to replace the εn in (1) by εn^t for $t > 1/2$. It is not known at present what could be done in this way for the finitely inhomogeneous random walk (see Problem 2 in Section 3, below). In the i.i.d. case, many such generalizations have been done. See for instance the work of Baum and Katz [5, 6], the work of Bai and Su [4], or the more recent results of Klesov [28] which appear to subsume many of the known results in the i.i.d. case (see also [39] which improves on Klesov's analogue of the result that (1) implies (2)). A similar type of generalization is to consider the summation in (1) along a subsequence of the positive integers; such work has been done by, e.g., Asmussen and Kurtz [2], Gut [20], and Klesov [28]. We should also mention Gut's extension [19] of Baum and Katz's work to the case of multidimensional indices, and note the work of Klesov [27, 28] and Deng [12] who extended Gut's results. We also remark that extensions of the Hsu-Robbins-Erdős result have been made to the case of randomly indexed sums; see Bai and Su [4], Adler [1], and Kuczmaszewska and Szynal [32].

Yet another type of generalization consists in allowing the random variables to take values in a Banach space; this kind of work is often connected with various geometric conditions on the Banach space, and is often also combined with extensions to non-identically distributed cases. See, e.g., [11, 23, 32, 35, 45, 46].

One may also generalize the Hsu-Robbins-Erdős law of large numbers by relaxing the independence of the random variables X_n. Work of this type, mainly under various mixing assumptions, was done for instance by Peligrad [37], Bingham [7, 8], Su [43], Shao [41], and Kong [29, 30]. Szynal [44] has proved that in the case of identically distributed and quadruplewise-independent random variables X_n, (2) and (3) still imply (1), and has shown via an example of Janson [24] that pairwise independence is insufficient for this implication (we note that nothing appears to be known about the converse implication here, and it also appears not to be known whether the quadruplewise independence can be relaxed to triplewise independence, even under the auxiliary hypothesis that $P(|X_n| \le 1) = 1$).

Finally, and this is the generalization that interests us most in the present paper, the Hsu-Robbins-Erdős theorem has been generalized in several ways to independent but non-identically distributed cases. In much of this kind of work, only analogues of Hsu and Robbins' result [22] that (2) and (3) imply (1) can be proved, and complete results giving the converse implication do not appear to be available. See for instance Woyczyński's notion of uniformly bounded tail probabilities [45, 46], the work of Duncan and Szynal [13] and its adaptation by Szynal [44] to the quadruplewise independent case, and finally the recent and quite general work of Li, Rao, Jiang, and Wang [35]. Also, many of the results of Klesov [28] can be extended to a generalization of the case of uniformly bounded tail probabilities [40]. Many of the proofs of Hsu-Robbins-Erdős type results, especially in non-identically distributed cases and including the proofs in the present paper, depend on a certain inequality of Hoffman-Jørgensen [21] (see, e.g., [35, Lemma 2.2]).

However, in the non-identically distributed cases, without extra structure it is difficult to get conditions for analogues of (1) which are both necessary and sufficient, and which depend only on the explicit knowledge of the sizes of the tail probabilities of the random variables. A generalization of the Hsu-Robbins-Erdős theorem with such a necessary and sufficient condition for the analogue of (1) has been obtained by Pruss [38] in the case of a triangular array of random variables having the extra structure of "regularly covering" a fixed distribution; this case generalized a partial result obtained by C. S. Kahane [25] for randomly sampled Riemann sums. (See also Problem 3 in Section 3, below.)

Finally, as we mentioned before, Spătaru [42] under some auxiliary conditions generalized the Hsu-Robbins-Erdős law of large numbers to an independent but not necessarily identically distributed case having the extra structure of a finitely inhomogeneous random walk; he also proved that no auxiliary conditions are needed to conclude that, still in the case of the finitely inhomogeneous random walk, (1) implies (2) and (3). Thus, the finitely inhomogeneous random walk provides us with a setting where under suitable auxiliary conditions we can prove that (2) and (3) are a necessary and sufficient condition for (1), analogously to original i.i.d. case of Hsu-Robbins-Erdős and to the regular covering generalization of [38]. The present paper is primarily concerned with the question of the auxiliary conditions under which such (2) and (3) imply (1). In light of Spătaru's converse result (which needs no auxiliary conditions), it will follow that under our auxiliary conditions (1) holds if and only if both (2) and (3) hold.

Henceforth, we suppose that the situation is as in the first paragraph of this paper, and thus that the set of distributions of the $\{X_n\}$ is finite. To state Spătaru's auxiliary conditions under which he proved that (2) and (3) imply (1), partition $\mathbb{Z}^+$ into disjoint sets $N_1, \dots, N_p$ such that if $n \in N_i$ then X_n has the same distribution as Y_i. To avoid triviality, assume that all the N_i are non-empty. Let $\alpha_i(n) = \operatorname{Card}\{k \in N_i\colon\ k \leq n\}$. Spătaru's auxiliary conditions are then as follows.

Condition **A**. For each $i \in \{1, \dots, p\}$ there exists $\theta_i \in [0, 1]$ and positive constants $C_1(i)$ and $C_2(i)$ such that

$$C_1(i)n^{\theta_i} \leq \alpha_i(n) \leq C_2(i)n^{\theta_i + 1/2},$$

for sufficiently large n.

Condition **B**. There is a constant C such that for any $i \in \{1, \dots, p\}$ we have

$$\sum_{k \in N_i \cap [n, \infty)} k^{-3} \leq C\frac{\alpha_i(n)}{n^3},$$

if n is sufficiently large.

Instead of the conjunction of these two conditions, we shall consider combinations of the following conditions.

Condition $\mathbf{V}_i$. There exists $\theta = \theta_i > 1$ such that

$$E\left[|Y_i|^\theta\right] < \infty.$$

Condition $\mathbf{W}_i$. There exists $\theta = \theta_i > 0$ such that

$$n^\theta \leq \alpha_i(n)$$

for n sufficiently large.

Condition $\mathbf{X}_i$. There exists $\theta = \theta_i < 1$ such that

$$\alpha_i(n) \leq n^\theta$$

for n sufficiently large.

Condition $\mathbf{Y}_i$. There is a finite number $C = C_i$ such that

$$\alpha_i(2n) \leq C\alpha_i(n)$$

for n sufficiently large.

Note that Condition $\mathbf{Y}_i$ is Feller's "dominated variation" [17] (see also [10, Sect. 1.10]).

Condition $\mathbf{Z}_i$. There exists $\theta = \theta_i > 0$ such that

$$\sum_{k \in N_i \cap [n, \infty)} k^{-\theta} \leq C \frac{\alpha_i(n)}{n^\theta},$$

for n sufficiently large.

Remarks. We can make the following elementary remarks concerning the relations between the various conditions:

(1) If $\mathbf{A}$ holds, then for every i we have $\mathbf{W}_i$ or $\mathbf{X}_i$ or both.

(2) $\mathbf{B}$ is equivalent to $\mathbf{Z}_i$ holding with $\theta_i = 3$ for every i.

(3) $\mathbf{Z}_i$ implies $\mathbf{Y}_i$ (Lemma 2, below).

(4) If (2) holds, then $\mathbf{W}_i$ implies $\mathbf{V}_i$ (Lemma 3, below).

(5) Following Klesov [28, Remark 4, p. 778], note that if $\mathbf{Z}_i$ is fulfilled for $\theta = \theta^{(0)}$, then it is also fulfilled for all $\theta > \theta^{(0)}$.

In light of the above remarks, the following theorem will be stronger than that of Spătaru [42], who, we recall, had required the truth of *both* Conditions $\mathbf{A}$ and $\mathbf{B}$. In particular, we will see that Spătaru's conclusions hold even if only $\mathbf{B}$ is assumed. On the other hand we will see later (see

Remarks following Theorem 2, below) that his Condition **A** is not sufficient by itself.

THEOREM 1. *Suppose that for every $i \in \{1, \ldots, p\}$ we have*

$$[(\mathbf{V}_i \text{ or } \mathbf{W}_i \text{ or } \mathbf{X}_i) \text{ and } \mathbf{Y}_i] \text{ or } \mathbf{Z}_i. \tag{4}$$

Then (1) *holds if and only if both* (2) *and* (3) *hold.*

Remarks. The proof of Theorem 1 will use work of Klesov [28] on rates of convergence in the law of large numbers for i.i.d. random variables. We do not know if (4) can be replaced by simply $\mathbf{Y}_i$. But we do know that $\mathbf{Y}_i$ cannot be avoided even in the case where $Y_2 \equiv 0$ and $p = 2$.

We say that a random variable A is *symmetric* if A and $-A$ have the same distribution.

THEOREM 2. *Suppose that $p = 2$ and that N_1 is such that $\mathbf{Y}_1$ fails. Then there exists a symmetric random variable Y_1 such that if X_n has the same distribution as Y_1 for $n \in N_1$ while $X_n \equiv 0$ for $n \notin N_1$, then both* (2) *and* (3) *hold while* (1) *fails. Moreover, there does exist a choice of N_1 such that $\mathbf{Y}_1$ fails while $\mathbf{W}_1$ holds. Indeed, this N_1 may even be chosen so that for* **every** $0 \le \theta < 1$ *we have*

$$n^{\theta} \le \alpha_1(n) \tag{5}$$

for n sufficiently large, and there exists a constant $C \in (0, 1)$ such that

$$Cn \le \alpha_2(n) \tag{6}$$

for n sufficiently large, where α_2 is defined in terms of $N_2 \stackrel{\text{def}}{=} \mathbb{Z}^+ \setminus N_1$.

Remarks. This gives a counterexample to Spătaru's conjecture that a result like Theorem 1 holds with no auxiliary conditions on the N_i or Y_i. Indeed, we see that Spătaru's Condition **A** (which is satisfied for $p = 2$ if (5) and (6) hold) is not sufficient by itself for Theorem 1.

But perhaps the question that Spătaru was studying was not quite the natural one. We can prove the following result.

THEOREM 3. *Suppose that for every $i \in \{1, \ldots, p\}$ we have*

$$\mathbf{V}_i \quad \text{or} \quad \mathbf{W}_i \quad \text{or} \quad \mathbf{X}_i \quad \text{or} \quad \mathbf{Z}_i. \tag{7}$$

Assume that for some $\varepsilon > 0$ *we have*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} P\big(|X_k| \geq \varepsilon n\big) < \infty \tag{8}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E\Big[X_k 1_{\{|X_k| < \varepsilon n\}}\Big] = 0. \tag{9}$$

Then,

$$\sum_{n=1}^{\infty} P\big(|S_n| \geq \varepsilon' n\big) < \infty, \tag{10}$$

where $\varepsilon' = c\varepsilon$ *for some constant* $c > 0$ *depending only on the choice of the* N_i, *on* p, *and on the various exponents involved in the conditions in* (7). *Conversely* (*without any auxiliary conditions on the* N_i *or* Y_i) *if* (10) *holds for some* $\varepsilon' > 0$ *then both* (8) *and* (9) *hold for* $\varepsilon = c'\varepsilon'$, *where* $c' > 0$ *is a constant depending on the* N_i *and on* p.

COROLLARY 1. *Suppose that for every* $i \in \{1, \dots, p\}$ *we have* (7) *holding. Then* (1) *holds if and only if for every* $\varepsilon > 0$ *both* (8) *and* (9) *hold.*

It is not known if this is true if the assumption of (7) is dropped. It seems to the author that the question considered in Corollary 1 is more natural than the one considered in Theorem 1.

To clarify the exact impact of Condition $\mathbf{Y}_i$, we formulate the following result.

THEOREM 4. *The following two conditions on the* N_i, $i = 1, \dots, p$, *are equivalent*:

(I) *for every* $i \in \{1, \dots, p\}$ *we have* $\mathbf{Y}_i$

(II) *whenever* $Y_1, \dots, Y_p$ *are random variables and the* X_n *are such that* X_n *has the same distribution as* Y_i *for* $n \in N_i$, *then* (2) *holds if and only if for every* $\varepsilon > 0$ *condition* (8) *holds.*

If $p = 2$, *the following two conditions on* N_1 *are equivalent*:

(i) $\mathbf{Y}_1$ *holds*

(ii) *whenever* Y_1 *is a random variable and the* X_n *are such that* X_n *has the same distribution as* Y_1 *for* $n \in N_1$ *while* $X_n \equiv 0$ *for* $n \notin N_1$, *then* (2) *holds if and only if for every* $\varepsilon > 0$ *condition* (8) *holds.*

Moreover, if (I) *or* (i), *respectively, fails then there is a counterexample for* (II) *or* (ii), *respectively, with all the* Y_i *symmetric.*

The assertion that (ii) implies (i) is equivalent to the first part of Theorem 2. A distinction similar to that between (2) and the validity of (8) for every $\varepsilon > 0$ is also discussed by Bingham and Goldie [9, Sect. 4.4].

2. PROOFS AND FURTHER DISCUSSION

The basic technique in the proofs is to split the sum up over the N_i. Write

$$S_n^{(i)} = \sum_{k \in [1,n] \cap N_i} X_k,$$

for $i \in \{1, \dots, p\}$. Then $S_n^{(i)}$ is a sum of i.i.d. random variables, and

$$S_n = S_n^{(1)} + \cdots + S_n^{(p)}.$$

Moreover, $S_n^{(1)}, \dots, S_n^{(p)}$ are independent for each fixed n.

Note that if (2) holds then each of the X_n and Y_i has finite expectation (see [42, Proof of Lemma 1]).

Now, given a random variable A, let $\mu(A)$ denote any of its medians, and let A^s denote its *symmetrization* $A - A'$, where A' is a random variable with the same distribution as A but independent of A. In the present paper, we shall define our symmetrizations so that we might have the relation $(A_1 + \cdots + A_n)^s = A_1^s + \cdots + A_n^s$ whenever we need it for independent random variables A_i, and we shall feel free to use this relation without any further explicit mention.

The following result can be construed as relating (1) to more classical questions concerning sums of i.i.d. random variables, since $S_n^{(i)}$ is a sum of precisely $\alpha_i(n)$ i.i.d. random variables.

PROPOSITION 1. *Suppose that all the Y_i have finite expectation. Assume that*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} E[X_k] = 0. \tag{11}$$

Then the following statements are equivalent:

(i) $\sum_{n=1}^{\infty} P(|S_n^{(i)} - E[S_n^{(i)}]| \geq \varepsilon n) < \infty$, *for every* $\varepsilon > 0$ *and each* $i \in \{1, \dots, p\}$

(ii) $\sum_{n=1}^{\infty} P(|S_n^{(i)} - \mu(S_n^{(i)})| \geq \varepsilon n) < \infty$ *for every* $\varepsilon > 0$ *and each* $i \in \{1, \dots, p\}$

(iii) $\sum_{n=1}^{\infty} P(|(S_n^{(i)})^s| \geq \varepsilon n) < \infty$, *for every* $\varepsilon > 0$ *and each* $i \in \{1, \ldots, p\}$

(iv) *condition* (1) *holds, i.e.,* S_n/n *converges completely to* 0 *as* $n \to \infty$.

Moreover, conditions (i)–(iii) *are also equivalent for every* **fixed** $i \in \{1, \ldots, p\}$.

Proof. First consider (i)–(iii) for a fixed i. If α_i is a bounded sequence (i.e., if N_i is finite), and $N_i \neq \varnothing$ (and we *did* assume that $N_i \neq \varnothing$) then it is easy to see that (i)–(iii) are equivalent to the assertion that Y_i has finite expectation, which again we have assumed. Now, suppose α_i is unbounded. The equivalence of (ii) and (iii) for fixed i follows from standard symmetrization inequalities (see, e.g., [36, Sect. 17.1.A]). Now, the weak law of large numbers implies that

$$\frac{\mu\left(S_n^{(i)}\right) - E\left[S_n^{(i)}\right]}{\alpha_i(n)} \to 0$$

as $n \to \infty$, since $S_n^{(i)}$ is the sum of $\alpha_i(n)$ i.i.d. random variables with finite expectation and since $\alpha_i(n) \to \infty$. Moreover, $\alpha_i(n) \leq n$ so that $(\mu(S_n^{(i)}) - E[S_n^{(i)}])/n \to 0$ and the equivalence of (i) and (ii) for fixed i follows.

We now show that (i) implies (iv). For, if $|S_n - E[S_n]| \geq \varepsilon n$, then it follows that for some $i \in \{1, \ldots, p\}$ we have $|S_n^{(i)} - E[S_n^{(i)}]| \geq \varepsilon n/p$. Hence,

$$P\left(|S_n - E[S_n]| \geq \varepsilon n\right) \leq \sum_{i=1}^{p} P\left(\left|S_n^{(i)} - E\left[S_n^{(i)}\right]\right| \geq \varepsilon n/p\right).$$

Since p is a fixed constant, by (i) it then follows that

$$\sum_{n=1}^{\infty} P\left(|S_n - E[S_n]| \geq \varepsilon n\right) < \infty,$$

for every $\varepsilon > 0$. But (11) implies that $E[S_n]/n \to 0$ so that (1) easily follows.

Finally, we show that (iv) implies (iii). To this end, recall that

$$P\left(\max_{1 \leq i \leq p} |T_i| \geq t\right) \geq \frac{\sum_{i=1}^{p} P\left(|T_i| \geq t\right)}{1 + \sum_{i=1}^{p} P\left(|T_i| \geq t\right)} \tag{12}$$

for every $t \geq 0$ whenever the T_i are independent random variables. This inequality can be found in [18, Proof of Lemma 3.2] or [34, Lemma 2.6].

Now, by an equality of Lévy type [33, Proposition 1.1.2], if the T_i are also symmetric then it follows that

$$2P\left(\left|\sum_{i=1}^{p} T_i\right| \geq t\right) \geq \frac{\sum_{i=1}^{p} P\left(|T_i| \geq t\right)}{1 + \sum_{i=1}^{p} P\left(|T_i| \geq t\right)}.$$

This useful inequality as well as (12) was kindly pointed out to the author by the anonymous referee of another one of the author's papers, and the author is most grateful for this. Letting $T_i = (S_n^{(i)})^s$, setting $t = \varepsilon n$, and remarking that all probabilities are bounded above by 1, we see that

$$\begin{aligned} 2(p+1)P\left(|S_n^s| \geq \varepsilon n\right) &= 2(p+1)P\left(\left|\sum_{i=1}^{p} \left(S_n^{(i)}\right)^s\right| \geq \varepsilon n\right) \\ &\geq \sum_{i=1}^{p} P\left(\left|\left(S_n^{(i)}\right)^s\right| \geq \varepsilon n\right). \end{aligned}$$

Then (iii) follows immediately from (1) and standard symmetrization inequalities. ∎

LEMMA 1. *Suppose that all the Y_i have finite mean. Then* (3) *and* (11) *are equivalent.*

Proof. Fix $\varepsilon > 0$. Choose N sufficiently large that for every $i \in \{1, \ldots, p\}$ and for each $n \geq N$ we have $|E[Y_i 1_{\{|Y_i| < n\}}] - E[Y_i]| \leq \varepsilon$. Then, for $n \geq N$ we will have

$$\left|\frac{1}{n}\sum_{k=1}^{n} E\left[X_k 1_{\{|X_k| < n\}}\right] - \frac{1}{n}\sum_{k=1}^{n} E[X_k]\right| \leq \varepsilon,$$

since the distributions of the X_n are chosen from among the distributions of the Y_i. Then, since n is arbitrary it follows that the left hand side of the above inequality tends to zero with n, so that (3) and (11) must indeed be equivalent. ∎

The following two lemmas were already alluded to in Section 1 of this paper.

LEMMA 2. *If N_i is such that $\mathbf{Z}_i$ holds, then $\mathbf{Y}_i$ holds likewise.*

Proof. In this proof, we suppress the subscripted i's for convenience. Let θ be as in the definition of $\mathbf{Z}_i$. Then,

$$\alpha(2n) - \alpha(n) \le (2n)^{\theta} \sum_{k \in N \cap [n, \infty)} k^{-\theta} \le (2n)^{\theta} C \frac{\alpha(n)}{n^{\theta}} = 2^{\theta} C \alpha(n),$$

by $\mathbf{Z}_i$, and so $\mathbf{Y}_i$ follows. ∎

LEMMA 3. *If N_i is such that $\mathbf{W}_i$ holds and* (2) *holds, then $\mathbf{V}_i$ likewise holds.*

Proof. By a simple rearrangement we can rewrite (2) in the form

$$\sum_{j=1}^{p} \sum_{n=1}^{\infty} \alpha_j(n) P\big(|Y_j| \ge n\big) < \infty. \tag{13}$$

Hence,

$$\sum_{n=1}^{\infty} \alpha_i(n) P\big(|Y_i| \ge n\big) < \infty.$$

By $\mathbf{W}_i$, there is a $\phi > 0$ such that $\alpha_i(n) \ge n^{\phi}$ for n sufficiently large. Thus,

$$\sum_{n=1}^{\infty} n^{\phi} P\big(|Y_i| \ge n\big) < \infty.$$

But this last inequality is equivalent to saying that $E[|Y_i|^{1+\phi}] < \infty$, and $\mathbf{V}_i$ follows upon letting $\theta = 1 + \phi$. ∎

Now let $A_i(n) = \sum_{k=1}^{n} \alpha_i(k)$.

LEMMA 4. *Suppose that $\mathbf{Z}_i$ holds. Then,*

$$\sum_{k=n}^{\infty} \frac{\alpha_i(k)}{k^{\theta+1}} = O\left(\frac{A_i(n)}{n^{\theta+1}}\right),$$

as $n \to \infty$, where θ is the exponent in $\mathbf{Z}_i$.

Proof. We shall drop the subscript i throughout the proof. Let 1_N be the indicator function of the set $N = N_i$. By Fubini's theorem we then have

$$\begin{aligned}
\sum_{k=n}^{\infty} \frac{\alpha(k)}{k^{\theta+1}} &= \sum_{k=n}^{\infty} \frac{\sum_{m=1}^{k} 1_N(m)}{k^{\theta+1}} \\
&= \sum_{m=1}^{\infty} \sum_{k=\max(n,m)}^{\infty} \frac{1_N(m)}{k^{\theta+1}} \\
&= \sum_{m\in N} \sum_{k=\max(n,m)}^{\infty} k^{-\theta-1} \\
&= O\Bigg(\sum_{m\in N} (\max(n,m))^{-\theta} \Bigg) \\
&= O\Bigg(\sum_{m\in N\cap[1,n]} n^{-\theta} \Bigg) + O\Bigg(\sum_{m\in N\cap(n,\infty)} m^{-\theta} \Bigg) \\
&= O\big(n^{-\theta}\alpha(n)\big) + O\big(n^{-\theta}\alpha(n)\big),
\end{aligned}$$

where in the last line we have used the definition of $\alpha(n)$ as well as Condition $\mathbf{Z}_i$. It remains to show that

$$n^{-\theta}\alpha(n) = O\left(\frac{A_i(n)}{n^{\theta+1}} \right).$$

To prove this, note that by Lemma 2 we have $\mathbf{Y}_i$, and by applying $\mathbf{Y}_i$ twice we see that $\alpha(4m) = O(\alpha(m))$. Suppose $n \ge 2$. Then, there is a natural number $m \le n/2$ such that $4m \ge n$, so that if $\lfloor n/2 \rfloor$ indicates the greatest integer not exceeding $n/2$ then

$$\alpha(n) \le \alpha(4m) = O(\alpha(m)) \le O(\alpha(\lfloor n/2 \rfloor)).$$

Moreover, $(n - \lfloor n/2 \rfloor)\alpha(\lfloor n/2 \rfloor) \le A(n)$ as can be readily seen from the monotonicity of α. Hence,

$$\begin{aligned}
n^{-\theta}\alpha(n) &= O\big(n^{-\theta}\alpha(\lfloor n/2 \rfloor)\big) \\
&\le O\big(n^{-\theta}(n - \lfloor n/2 \rfloor)^{-1} A(n)\big) \\
&\le O\big(n^{-\theta} n^{-1} \cdot 2 \cdot A(n)\big) \\
&= O\big(n^{-\theta-1} A(n)\big),
\end{aligned}$$

as desired. ∎

Finally, we need the following modified version of a special case of Klesov [28, Lemma 5]. The proof of our result uses an inequality of Hoffman-Jørgensen [21] (see [35, Lemma 2.2]) and an inequality of von Bahr and Esseen [3], and is almost identical to Klesov's proof of his [28, Lemma 5] so that we omit it.

LEMMA 5. *Suppose that T_n is the sum of $\alpha(n)$ independent copies of a symmetric random variable X. Suppose that for every $\varepsilon > 0$ we have*

$$\sum_{n=1}^{\infty} \alpha(n) P(|X| \geq \varepsilon n) < \infty,$$

and that there is a $\lambda \geq 1$ and a $t \in (0,2]$ such that

$$\sum_{n=1}^{\infty} \left(\frac{\alpha(n) E\left[|X|^t 1_{\{|X|<n\}}\right]}{n^t} \right)^{\lambda} < \infty. \tag{14}$$

Then

$$\sum_{n=1}^{\infty} P\big(|T_n| \geq \varepsilon n\big) < \infty,$$

for every $\varepsilon > 0$.

I now claim that Theorem 4 together with the above work and the methods of Klesov [28] implies Theorem 1.

Proof of Claim. The fact that (1) implies (2) and (3) was already proved by Spătaru [42]. Conversely, assume that (2) and (3) hold. In light of Proposition 1 and Lemma 1, it suffices to show that for every $i \in \{1, \ldots, p\}$ and every $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} P\Big(\Big|\big(S_n^{(i)}\big)^s\Big| \geq \varepsilon n\Big) < \infty. \tag{15}$$

Fix $i \in \{1, \ldots, p\}$. Since we know that $\mathbf{Z}_i$ implies $\mathbf{Y}_i$ by Lemma 2 and we do have $\mathbf{Z}_i$ or $\mathbf{Y}_i$, it follows by Theorem 4 that (8) must hold for every $\varepsilon > 0$. Rearranging (8), we see that for every $\varepsilon > 0$,

$$\sum_{j=1}^{p} \sum_{n=1}^{\infty} \alpha_j(n) P\big(|Y_j| \geq \varepsilon n\big) < \infty. \tag{16}$$

Hence, in particular

$$\sum_{n=1}^{\infty} \alpha_i(n) P\big(|Y_i| \geq \varepsilon n\big) < \infty,$$

for every $\varepsilon > 0$. By symmetrization inequalities [36, Sect. 17.1.A], we must also have

$$\sum_{n=1}^{\infty} \alpha_i(n) P\big(|Y_i^s| \geq \varepsilon n\big) < \infty, \tag{17}$$

for every $\varepsilon > 0$. We shall apply Lemma 5 to conclude that (15) holds.

If $\mathbf{W}_i$ holds, then $\mathbf{V}_i$ holds likewise by Lemma 3. Now, if $\mathbf{V}_i$ holds then we have $E[|Y_i|^\theta] < \infty$ for some $\theta > 1$. Without loss of generality assume $\theta \leq 2$. But if $E[|Y_i|^\theta] < \infty$ then $E[|Y_i^s|^\theta] < \infty$ likewise by symmetrization inequalities, and (14) with $X = Y_i^s$ follows as soon as we choose any $\lambda > 1/(\theta - 1)$ and set $t = \theta$, since $\alpha_i(n) \leq n$, and so (15) indeed follows from Lemma 5 if $\mathbf{V}_i$ or $\mathbf{W}_i$ holds. This case also follows from Li, Rao, Jiang, and Wang [35, Theorem 2.1] (see [35, Remark 2.2]). Note that [35, Theorem 2.1] also uses the same inequality of Hoffman-Jørgensen that is involved in the proof of Lemma 5.

If $\mathbf{X}_i$ holds then choose $\lambda > (1 - \theta)^{-1}$. Then (14) holds by $\mathbf{X}_i$ if we set $X = Y_i^s$ and put $t = 1$, since $E[|X|] < \infty$ by symmetrization as we had already remarked that all the Y_j have finite mean if (2) holds. Once again, (15) follows from Lemma 5.

The remaining case is that of $\mathbf{Z}_i$ holding. But in that case, imitate the notation of Klesov [28] and set $X = Y_i^s$, $\tau_n = \alpha_i(n)/n$, $T_n = A_i(n)$, $\alpha = 1$ and let θ be as in the definition of $\mathbf{Z}_i$. Then, by Lemma 4 we have

$$\sum_{n=m}^{\infty} \frac{\tau_n}{n^\theta} = O\big(T_m / m^{\theta+1}\big),$$

as $m \to \infty$ so that Klesov [28, Proof of Theorem 4] tells us that (17) implies that

$$\sum_{n=1}^{\infty} \tau_n \Big(n^{1-t} E\Big[|X|^t 1_{\{|X| < n\}}\Big]\Big)^\lambda < \infty,$$

where $t = 2$ and λ is a natural number bigger than θ. Then, (14) follows immediately since $\alpha(n) \leq n$, and so we are done by Lemma 5. ∎

Remark. To prove Theorem 3, all one needs to do is to carefully go through the proof of Theorem 1 and realize that if one takes a greater care as to the ε multiplying the n in expressions containing things like $P(|\cdot| \geq \varepsilon n)$ or $E[\ldots 1_{\{|\cdot| \geq \varepsilon n\}}]$, then in all the statements where we (or one of our references) have passed from one such expression holding for all $\varepsilon > 0$ to another such expression holding for all $\varepsilon > 0$, we may in fact pass from the first holding for one specific ε to the second holding for $c\varepsilon$ where c is some easy-to-determine finite positive constant (differing from expression to expression). We leave the verification of this observation as an exercise for the reader. Much the same observation was used to produce another extension of Klesov's results in [39], where too the observation was left for the reader to verify.

It remains to give a proof of Theorem 4 and of the "moreover" in Theorem 2.

Proof of Theorem 4. In light of the equivalence of (2) and (13) as well as of (8) and (16), we split up the problem into separate questions for each i. Suppose first that $\mathbf{Y}_i$ holds and that

$$\sum_{n=1}^{\infty} \alpha_i(n) P\big(|Y_i| \geq n\big) < \infty. \tag{18}$$

I claim that then for every $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \alpha_i(n) P\big(|Y_i| \geq \varepsilon n\big) < \infty. \tag{19}$$

The fact that (I) implies (II) and that (i) implies (ii) easily follows from this claim. To prove the claim, fix $\varepsilon > 0$ and choose a natural number $\lambda \geq \varepsilon^{-1}$. Recall that α_i is non-decreasing. Then,

$$\begin{aligned}\sum_{n=\lambda}^{\infty} \alpha_i(n) P\big(|Y_i| \geq \varepsilon n\big) &\leq \sum_{n=\lambda}^{\infty} \alpha_i(n) P\big(|Y_i| \geq n/\lambda\big)\\ &= \sum_{k=0}^{\lambda-1} \sum_{m=1}^{\infty} \alpha_i(m\lambda + k) P\big(|Y_i| \geq (m\lambda + k)/\lambda\big)\\ &\leq \sum_{k=0}^{\lambda-1} \sum_{m=1}^{\infty} \alpha_i((m+1)\lambda) P\big(|Y_i| \geq m\big)\\ &= \lambda \sum_{m=1}^{\infty} \alpha_i((m+1)\lambda) P\big(|Y_i| \geq m\big). \end{aligned} \tag{20}$$

Also, $\alpha_i((m+1)\lambda) \leq \alpha_i(2\lambda m) \leq C^j \alpha_i(m)$ for sufficiently large m, where j is chosen so that $2^j \leq 2\lambda$ and C is the constant in the definition of $\mathbf{Y}_i$. Then, the convergence of the right hand side of (20) follows from (18), and (19) follows from the convergence of the left hand side of (20). This proves the claim.

Conversely, we shall simultaneously show that the negation of (I) implies the negation of (II) and that the negation of (i) implies the negation of (ii). For, suppose that for some $i \in \{1, \dots, p\}$ we have $\mathbf{Y}_i$ failing to hold. Let $Y_j \equiv 0$ for $j \neq i$. We shall exhibit a choice of Y_i which is symmetric and has (18) holding, but is such that (19) fails for $\varepsilon = 1/4$. The equivalence of

(2) and (13) and that of (8) and (16) will then complete the proof.

We now drop the subscript i for convenience. Recall that $A(n) = \sum_{k=1}^{n} \alpha(k)$. Then, the monotonicity of $\alpha(n)$ implies that

$$n\alpha(n) \geq A(n). \tag{21}$$

Since $\mathbf{Y}_i$ fails, choose a strictly increasing sequence of natural numbers λ_m such that $\alpha(2\lambda_m) \geq 2m\alpha(\lambda_m) > 0$. Note that certainly we will have $A(\lambda_m) > 0$ for every m. Let

$$B(N) = \sum_{n=1}^{N} \alpha(4n).$$

Note that

$$\operatorname{Card}\{4n: 4n \geq 2\lambda_m, 1 \leq n \leq \lambda_m\} \geq \lambda_m/2.$$

But if $4n \geq 2\lambda_m$, then $\alpha(4n) \geq \alpha(2\lambda_m) \geq 2m\alpha(\lambda_m)$ so that

$$B(\lambda_m) \geq (\lambda_m/2)(2m\alpha(\lambda_m)) \geq mA(\lambda_m), \tag{22}$$

by (21).

Let

$$\gamma(n) = \sum_{m=1}^{\infty} \frac{1_{[1,\lambda_m]}(n)}{m^2 A(\lambda_m)},$$

where $1_{[1,\lambda_m]}$ is of course the indicator function of the interval $[1, \lambda_m]$. Then γ is non-increasing since it is a sum of non-increasing functions. I claim that

$$\sum_{n=1}^{\infty} \alpha(n)\gamma(n) < \infty. \tag{23}$$

To see this, note that by Fubini's theorem we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \alpha(n)\gamma(n) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1_{[1,\lambda_m]}(n)\alpha(n)}{m^2 A(\lambda_m)} \\
&= \sum_{m=1}^{\infty} \frac{\sum_{n=1}^{\lambda_m} \alpha(n)}{m^2 A(\lambda_m)} \\
&= \sum_{m=1}^{\infty} \frac{A(\lambda_m)}{m^2 A(\lambda_m)} = \sum_{m=1}^{\infty} m^{-2} < \infty.
\end{aligned}$$

Now, we may easily choose a symmetric random variable Y such that $P(|Y| \geq n) = \gamma(n)/\gamma(1)$ for $n \geq 1$ since $\gamma(n)/\gamma(1)$ is non-increasing and bounded above by 1. By (23), $Y = Y_i$ will satisfy (18). We now show that it will fail to satisfy (19) if $\varepsilon = 1/4$. We may use Fubini's theorem as well as (22) to see that

$$\begin{aligned}
\sum_{n=1}^{\infty} \alpha(n) P(|Y| \geq n/4) &\geq \sum_{n \in 4\mathbb{Z}^+} \alpha(n) P(|Y| \geq n/4) \\
&= \sum_{n=1}^{\infty} \alpha(4n) P(|Y| \geq n) \\
&= \sum_{n=1}^{\infty} \alpha(4n) \gamma(n)/\gamma(1) \\
&= (\gamma(1))^{-1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1_{[1, \lambda_m]}(n) \alpha(4n)}{m^2 A(\lambda_m)} \\
&= (\gamma(1))^{-1} \sum_{m=1}^{\infty} \frac{B(\lambda_m)}{m^2 A(\lambda_m)} \\
&\geq (\gamma(1))^{-1} \sum_{m=1}^{\infty} \frac{mA(\lambda_m)}{m^2 A(\lambda_m)} \\
&= (\gamma(1))^{-1} \sum_{m=1}^{\infty} m^{-1} = \infty.
\end{aligned}$$

Hence, indeed, (19) fails for $\varepsilon = 1/4$. ∎

It only remains to prove the "moreover" of Theorem 2, since the rest of Theorem 2 follows from Theorem 4.

Proof of "Moreover" of Theorem 2. We shall define a set N_1 with the desired properties. Our example is inspired by one of Krasnosel'skii and Rutickii [31, p. 28]. Let

$$N_1 = \left(\bigcup_{n=2}^{\infty} (n!/2, n!] \right) \cap \mathbb{Z}^+.$$

I claim that N_1 has the requisite properties. If $\alpha_1(m) = \text{Card}\{k \in N_1: k \leq m\}$, then clearly $\alpha_1(n!) \geq n!/2$ for $n \geq 2$. But, if $n > 2$ then we easily see that

$$\begin{aligned}
\alpha_1(n!/2) &= \alpha_1((n-1)!) = 2!/2 + 3!/2 + \cdots + (n-1)!/2 \\
&= O((n-1)!), \qquad (24)
\end{aligned}$$

as $n \to \infty$. Hence, $\alpha_1(n!)/\alpha_1(n!/2) \to \infty$ as $n \to \infty$ and $\mathbf{Y}_1$ cannot hold. Now, if $n! \le m < (n+1)!$ for $n \ge 2$, then $\alpha_1(m) \ge \alpha_1(n!) \ge n!/2$. Fix $\theta \in (0,1)$. Then,

$$\begin{aligned} m^{-\theta}\alpha_1(m) &> ((n+1)!)^{-\theta} n!/2 \\ &= (n!)^{1-\theta}(n+1)^{-\theta}/2 \to \infty \end{aligned}$$

as $m \to \infty$ (or equivalently as $n \to \infty$). Hence indeed $m^{\theta} = o(\alpha_1(m))$ as $m \to \infty$, and (5) follows. On the other hand, if $n! \le m \le (n+1)!/2$ and $n \ge 2$ then

$$\alpha_1(m) = n!/2 + \alpha_1(n!/2) = n!/2 + o(n!) \le m/2 + o(m), \quad (25)$$

by (24), while if $(n+1)!/2 \le m \le (n+1)!$ then we also have

$$\begin{aligned} \alpha_1(m) &= m - (n+1)!/2 + \alpha_1(n!) < m - (m/2) + n! \\ &= m/2 + o(m), \end{aligned} \quad (26)$$

for $n \ge 2$. Fix any $C \in (0, 1/2)$. Then (6) follows immediately from (25), (26), and the identity $\alpha_2(m) = m - \alpha_1(m)$. ∎

3. FINAL REMARKS AND SOME PROBLEMS

First for the reader's convenience we recall the following question raised in Section 1.

Open Problem 1. Determine whether the assumption that $\mathbf{Y}_i$ holds for every $i \in \{1, \dots, p\}$ is sufficient so that (2) and (3) might imply (1). Determine whether assumption (7) can be dropped out of Theorem 3 and Corollary 1.

If (7) could be dropped out of Theorem 3 or even just out of Corollary 1, then by Theorem 4, indeed $\mathbf{Y}_i$ would be sufficient so that (2) and (3) might imply (1).

For a different type of problem, note that the method of the present paper can perhaps also be used to consider conditions for the convergence of series of the form

$$\sum_{n=1}^{\infty} \tau_n P\big(|S_n| \ge \varepsilon n^t\big) \quad (27)$$

for every $\varepsilon > 0$, where t is a fixed real number in $(1/2, \infty)$. Such questions are considered in the i.i.d. case by Klesov [28], and it is likely that a number of positive results about such things can also be proved in the case of the finitely inhomogeneous random walk S_n considered in the present paper.

Open Problem 2. Formulate and prove analogues of the results of Klesov [28] concerning the convergence of (27), perhaps by following the methods which were used in the present paper to study (1).

We recall that a quite different generalization of the Hsu-Robbins-Erdős law of large numbers to a non-identically distributed case ("regular covering") was considered in [38].

Open Problem 3. Find a common framework that subsumes both Theorem 1 of the present paper and the results of [38].

The main difficulty would not appear to rest as much in proving more general analogues of (2) and (3) implying (1) under some conditions, but rather it seems to rest in proving more general analogues of the converse result that (1) implies both (2) and (3). Perhaps this difficulty could be overcome by using some combination of the methods of [38, 39]. Finally, we remark that the last section of [39] raises a problem which is connected to the Hsu-Robbins-Erdős law of large numbers in the non-identically distributed case of [38], and which would perhaps have to be settled if the common framework of an answer to Problem 3 were to also subsume questions of convergence of series such as (27).

Finally we have the following question.

Open Problem 4. How much of Theorem 1 would still hold if the assumption of independence were weakened to quadruplewise independence?

The work of Szynal [44] is quite relevant to this question.

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